

COLIMITS OF MONADS

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Dedicated to the seventieth birthday of Manuela Sobral.

ABSTRACT. The category of all monads over many-sorted sets (and over other "set-like" categories) is proved to have coequalizers and strong coin-
 intersections. And a general diagram has a colimit whenever all the monads
 involved preserve monomorphisms and have arbitrarily large joint pre-
 fixpoints. In contrast, coequalizers fail to exist e.g. for monads over the
 (presheaf) category of graphs.

For more general categories we extend the results on coproducts of
 monads from [2]. We call a monad separated if, when restricted to monomor-
 phisms, its unit has a complement. We prove that every collection of sep-
 arated monads with arbitrarily large joint pre-fixpoints has a coproduct.
 And a concrete formula for these coproducts is presented.

1. INTRODUCTION

Whereas limits in the category $\text{Monad}(\mathcal{A})$ of monads over a complete cat-
 egory \mathcal{A} are easy, since the forgetful functor into the category $[\mathcal{A}, \mathcal{A}]$ of all
 endofunctors creates limits, colimits are more interesting. For example, a co-
 product of two monads need not exist in $\text{Monad}(\mathcal{A})$ – in fact, there are only
 four (trivial) types of monads over \mathbf{Set} having a coproduct with every monad,
 as proved in [2], see Theorem 4.4 below. In that paper a formula for coproducts
 of monads over \mathbf{Set} was presented, and we extend it to coproducts of separated
 monads over general categories \mathcal{A} . Separatedness means that a complement of
 the unit of the monad exists if we restrict ourselves to the category \mathcal{A}_m of ob-
 jects and monomorphisms of \mathcal{A} . All consistent monads over \mathbf{Set} are separated,
 see [2]. For other base categories many interesting monads fail to be separated.

Our main result is that in "set-like" categories, e. g., many-sorted sets, vec-
 tor spaces or sets and partial functions, the category $\text{Monad}(\mathcal{A})$ has (a) all
 coequalizers and strong cointersections and (b) colimits of every diagram of
 monos-preserving monads with arbitrarily large joint pre-fixpoints. (An object

X is a pre-fixpoint of a monad S if SX is a subobject of X .) That last condition is proved to be weaker than assuming that the monads are accessible. Moreover, arbitrarily large joint pre-fixpoints are sufficient for coproducts of

- (1) monos-preserving monads over set-like categories
- (2) separated monads over rather general categories.

And if $\mathcal{A} = \mathbf{Set}$, this condition is in case of coproducts of consistent monads also necessary (unless all but one of the monads are of the trivial type, see Theorem 4.4 below). It is an open problem whether having arbitrarily large joint pre-fixpoints is sufficient for coproducts of general monads over "reasonably" general categories.

Colimits of monads were studied by Kelly [9] who proved, inter alia, that for locally presentable base categories \mathcal{A} every diagram of accessible monads has a colimit in $\mathbf{Monad}(\mathcal{A})$. Kelly also proved a formula for the colimit. In case of coproducts of consistent monads over \mathbf{Set} a much simpler formula was presented in [2], inspired by the work of Ghani and Ustalu [8]: let S and T be consistent λ -accessible monads with unit complements \bar{S} and \bar{T} , respectively. Then the coproduct monad is given by

$$A \mapsto A + \operatorname{colim}_{i < \lambda} X_i + \operatorname{colim}_{i < \lambda} Y_i$$

Here X_i and Y_i are the λ -chains formed by colimits on limit ordinals, whereas the isolated steps are defined by the following mutual recursion:

$$X_{i+1} = \bar{S}(Y_i + A) \text{ and } Y_{i+1} = \bar{T}(X_i + A)$$

We prove that, unsurprisingly, the same formula holds for coproducts of separated monads on general categories.

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2. THE CATEGORY OF MONADS

In this section some basic properties of the category of monads and monad morphisms are collected.

Notation 2.1. (a) *Given a category \mathcal{A} we write $[\mathcal{A}, \mathcal{A}]$ for the category of endofunctors on \mathcal{A} and natural transformations between them. And $\mathbf{Monad}(\mathcal{A})$ denotes the category of monads and monad morphisms. The obvious forgetful functor is denoted by $V : \mathbf{Monad}(\mathcal{A}) \rightarrow [\mathcal{A}, \mathcal{A}]$.*

(b) We use $*$ to denote the parallel (horizontal) composition of natural transformations: given $a : F \rightarrow F'$ and $b : G \rightarrow G'$, where all functors are endofunctors of \mathcal{A} , we have $a*b : FG \rightarrow F'G'$ given by $a*b = aG' \cdot Fb = F'b \cdot aG$. Recall also the interchange law:

$$(c * d) \cdot (a * b) = (c \cdot a) * (d \cdot b).$$

Proposition 2.2. *The forgetful functor of $\text{Monad}(\mathcal{A})$ creates limits.*

Remark. Recall that creation of limits means that for every diagram D in $\text{Monad}(\mathcal{A})$ with a limit cone $p_d : T \rightarrow W D d$ of the underlying diagram in $[\mathcal{A}, \mathcal{A}]$ there exists a unique structure of a monad on T for which each p_d is a monad morphism. Moreover, the resulting cone is a limit in $\text{Monad}(\mathcal{A})$.

PROOF. For the given diagram

$$D : \mathcal{D} \rightarrow \text{Monad}(\mathcal{A})$$

denote the objects by

$$Dd = (T_d, \mu_d, \eta_d) \quad (d \in \text{obj}\mathcal{D}).$$

Given a limit cone $p_d : T \rightarrow T_d$, the unit of the monad on T is, necessarily, the unique natural transformation

$$\eta^T : Id \rightarrow T \text{ with } p_d \cdot \eta^T = \eta_d \quad (d \in \text{obj}\mathcal{D}).$$

(Recall that p_d 's are required to preserve unit.) And the multiplication $\mu^T : T \cdot T \rightarrow T$ is, necessarily, the unique natural transformation for which the squares

$$\begin{array}{ccc} T \cdot T & \xrightarrow{\mu^T} & T \\ p_d * p_d \downarrow & & \downarrow p_d \\ T_d \cdot T_d & \xrightarrow{\mu_d} & T_d \end{array}$$

commute for all $d \in \text{obj}\mathcal{D}$. The verification of the monad axioms is easy. To verify that this is a limit cone, let $q_d : (S, \mu^S, \eta^S) \rightarrow (T_d, \mu_d, \eta_d)$ be a cone of D . There exists a unique natural transformation $g : S \rightarrow T$ with $q_d = p_d \cdot q$ ($d \in \text{obj}\mathcal{D}$). It is a monad morphism. Indeed, the axiom $q \cdot \eta^S = q^T$ follows, since (p_d) is a moncone, from

$$p_d \cdot (q \cdot \eta^S) = q_d \cdot \eta^S = \eta_d = p_d \cdot \eta^T.$$

Analogously, the axiom $q \cdot \mu^S = \mu^T \cdot q * q$ follows from

$$p_d \cdot (\mu^T \cdot (q * q)) = \mu_d \cdot (p_d * p_d) \cdot (q * q) = p_d \cdot \mu^T \cdot (q * q)$$

□

Corollary 2.3. *Limits of monads over a complete category \mathcal{A} are computed object-wise (on the level of \mathcal{A}).*

Proposition 2.4. *The forgetful functor of $\text{Monad}(\mathcal{A})$ creates absolute coequalizers.*

Remark. Recall that this means that given a parallel pair of monad morphisms $p, q : S \rightarrow T$ whose coequalizers in $[\mathcal{A}, \mathcal{A}]$

$$S \begin{array}{c} \xrightarrow{p} \\ \xrightarrow{q} \end{array} T \xrightarrow{c} C$$

is absolute (that is, preserved by every functor with domain $[\mathcal{A}, \mathcal{A}]$), there exists a unique monad structure on C making c a monad morphism. Moreover, c is a coequalizer of p and q in $\text{Monad}(\mathcal{A})$.

PROOF. The unit of C is, necessarily,

$$\eta^C = c \cdot \eta^T.$$

To define the multiplication $\mu^C : C \cdot C \rightarrow C$, use the endofunctor of $[\mathcal{A}, \mathcal{A}]$ defined by $X \mapsto X \cdot X$ on objects and by $f \mapsto f * f$ on morphisms. Since $c * c$ is the coequalizer of $p * p$ and $q * q$, we have a unique μ^C for which c preserves multiplication:

$$\begin{array}{ccccc} S \cdot S & \begin{array}{c} \xrightarrow{p * p} \\ \xrightarrow{q * q} \end{array} & T \cdot T & \xrightarrow{c * c} & C \cdot C \\ \mu^S \downarrow & & \mu^T \downarrow & & \downarrow \mu^C \\ S & \begin{array}{c} \xrightarrow{p} \\ \xrightarrow{q} \end{array} & T & \xrightarrow{c} & C \end{array}$$

The verification that (C, η^C, μ^C) is a monad and c is a coequalizer in $\text{Monad}(\mathcal{A})$ is easy. \square

Definition 2.5. *An object Z is a fixpoint of an endofunctor H if $HZ \simeq Z$, and it is a pre-fixpoint of H if HZ is a subobject of Z .*

We say that H has arbitrarily large pre-fixpoints provided that for every object X there exists a pre-fixpoint Z of H with $Z \simeq Z + X$.

Example 2.6. *A monos-preserving endofunctor H of the category Set^S of many-sorted sets has arbitrarily large pre-fixpoints iff for every cardinal α there exists a pre-fixpoint of H all components of which have at least α elements.*

Notation 2.7. (a) *For an endofunctor H of \mathcal{A} an algebra is a pair (A, a) consisting of an object A and a morphism $a : HA \rightarrow A$. Homomorphisms of algebras are defined by the usual commutative square. The resulting category is denoted by $\text{Alg}H$.*

- (b) μH denotes the initial algebra (if it exists). By Lambek's Lemma [10] its algebra structure is invertible, thus, μH is a fixpoint of H .
- (c) If H has free algebras, i. e., the forgetful functor $\mathbf{Alg} H \rightarrow \mathcal{A}$ has a left adjoint, then \mathbb{F}_H denotes the corresponding monad over \mathcal{A} . And $\hat{\eta} : \text{Id} \rightarrow \mathbb{F}_H$ denotes its unit, whose components are the universal arrows of the free algebras.

Lemma 2.8. *Every accessible endofunctor of a cocomplete category with monic coproduct injections has arbitrarily large pre-fixpoints.*

PROOF. If H is accessible, then every object B generates a free H -algebra \bar{B} and $\bar{B} = B + H\bar{B}$, see [1]. Given an object A let B be an infinite copower of A . Then the equality $A + B \simeq B$ implies $A + \bar{B} \simeq \bar{B}$, and \bar{B} is a pre-fixpoint. \square

Theorem 2.9 (Barr [6]). *If an endofunctor H has free algebras, then \mathbb{F}_H is a free monad on H . The converse holds whenever the base category is complete.*

Example 2.10. *The power-set functor \mathcal{P} has no fixpoint, hence, it does not generate a free monad.*

Construction 2.11 (see [1]). *For every object X of \mathcal{A} define the free-algebra chain $W : \mathbf{Ord} \rightarrow \mathcal{A}$ (with objects W_i and morphisms $w_{i,j} : W_i \rightarrow W_j$ for all ordinals $i \leq j$) uniquely up to natural isomorphism by the following transfinite induction:*

The objects are given by

$$\begin{aligned} W_0 &= X \\ W_{i+1} &= X + HW_i, \end{aligned}$$

and

$$W_j = \text{colim}_{i < j} W_i \text{ for limit ordinals } j.$$

The morphisms are as follows:

$$\begin{aligned} w_{0,1} &: X \rightarrow X + HX, \text{ coproduct injection} \\ w_{i+1,j+1} &= \text{id}_X + Hw_{i,j} \end{aligned}$$

and

$$(w_{i,j})_{i < j} \text{ is a colimit cocone (for limit ordinals } j).$$

Whenever this chain converges after i steps, i. e., all connecting maps $w_{i,j}$ are isomorphisms, then as proved in [1],

$$W_i = F_H X$$

is the free algebra on X . More detailed, the two components of

$$(w_{i,i+1})^{-1} : X + HW_i \rightarrow W_i$$

are the universal arrow and the algebra structure of W_i , respectively.

Definition 2.12. A cocomplete category is said to have stable monomorphisms if

- (a) coproducts of parallel collections of monomorphisms are monic and
- (b) colimits of chains of monomorphisms consist of monics, and the factorizing map of every cocone of monics is monic.

Example 2.13. Sets, graphs, posets, many-sorted sets and almost all "usual" varieties of algebras have stable monomorphisms. All presheaf categories have stable monomorphisms.

Condition (b) implies that the unique morphism from 0 to any given object is monic (since 0 is the colimit of the empty chain). Thus rings are an example of a variety not having stable monomorphisms. Indeed, the initial ring is the ring \mathbb{Z} of integers, and not all ring homomorphisms with this domain are monic.

Theorem 2.14 (See [13]). Let H be an endofunctor of a cocomplete category with stable monomorphisms. If H preserves monomorphisms, the following conditions are equivalent:

- (1) H has free algebras
- (2) for every object X the free-algebra chain converges and
- (3) for every object X there exists an object Z with

$$HZ + X \text{ a subobject of } Z.$$

Corollary 2.15. Let \mathcal{A} be a cocomplete category with stable monomorphisms. Every monos-preserving endofunctor with arbitrarily large pre-fixpoints generates a free monad.

Indeed, we verify Condition (3) above: choose a pre-fixpoint Z with $Z \simeq Z + X$ to get $HZ + X$ as a subobject of $Z + X \simeq Z$.

Remark 2.16. Under the assumptions of the above theorem the free monad \mathbb{F}_H preserves monomorphisms. Indeed, let $m : X \rightarrow X'$ be a monomorphism. Denote by W'_i the free-algebra chain above for X' . It is easy to see that we get a natural transformation

$$m_i : W_i \rightarrow W'_i \quad (i \in \mathbf{Ord})$$

by

$$\begin{aligned} m_0 &= m : X \rightarrow X' \\ m_{i+1} &= m + Hm_i : X + HW_i \rightarrow X' + HW'_i \end{aligned}$$

and

$$m_j = \operatorname{colim}_{i < j} m_i \text{ for limit ordinals } j.$$

An easy transfinite induction shows that m_i is monic for every i : in the isolated step use the preservation of monics by H .

We know from the above theorem that for some ordinal i we have

$$F_H X = W_i \text{ and } F_H X' = W'_i.$$

For this ordinal we then also have

$$F_H m = m_i$$

(which follows by an easy inspection of the proof of the above theorem). Thus, $F_H m$ is monic.

Remark 2.17. If free H -algebras exist, the free monad \mathbb{F}_H fulfils

$$F_H = H \cdot F_H + \operatorname{Id}, \text{ with } \hat{\eta} \text{ as the right-hand injection.}$$

Indeed, for every object X let \bar{X} the free algebra on $\hat{\eta}_X : X \rightarrow \bar{X}$ with the algebra structure $\varphi_X : H\bar{X} \rightarrow \bar{X}$. Then $\bar{X} = H\bar{X} + X$ since $[\varphi_X, \hat{\eta}_X] : H\bar{X} + X \rightarrow \bar{X}$ is an isomorphism. (This is Lambek's Lemma applied to $H(-) + X$.) Since $F_H X = \bar{X}$, we see that the natural transformations $\varphi : HF_H \rightarrow F_H$ and $\hat{\eta} : \operatorname{Id} \rightarrow F_H$ form coproduct injections of $F_H = H \cdot F_H + \operatorname{Id}$.

3. SET-LIKE BASE CATEGORIES

For the base categories \mathcal{A} such as

- Set or Set^S (many-sorted sets)
- K-Vec (vector spaces)
- Set_* (sets and partial functions)

we prove that the category of monads has coequalizers and cointersections. And it has colimits of every diagram of monos-preserving monads that posses arbitrarily large joint pre-fixpoints. In case of coproducts over $\mathcal{A} = \operatorname{Set}$ that last condition was proved to be "almost" necessary in [2]: a collection of nontrivial monads over Set has a coproduct iff they posses arbitrarily large joint fixpoints. We explain this in more detail in the next section devoted to coproducts of separated monads.

Assumptions 3.1. Throughout this section \mathcal{A} denotes a category which has

- (a) limits and colimits
 - (b) stable monomorphisms (see Definition 2.12)
- and
- (c) split epimorphisms.

- Remark 3.2.** (a) \mathcal{A} is cowellpowered: every object X has only a set of quotients because X has only a set of idempotent endomorphisms. Indeed, for every quotient $e : X \rightarrow Y$ choose a splitting $i : Y \rightarrow X$ and get an idempotent $i \cdot e$, then two epimorphisms with the same idempotent yield the same quotient.
- (b) \mathcal{A} has (strong epi, mono)-factorizations of morphisms since every cowellpowered, cocomplete category does, see [4], 15.17.

Lemma 3.3. Monad (\mathcal{A}) has (strong epi, mono)-factorization of morphisms, and every strong epimorphism has all components epic.

PROOF. We prove that every monad morphism $f : \mathbb{S} \rightarrow \mathbb{R}$ has a factorization $f = m \cdot e$ in $\text{Monad}(\mathcal{A})$ where m has monic components and e has (split) epic ones. It follows easily from Proposition 2.2 that m is a monomorphism in $\text{Monad}(\mathcal{A})$ and e is a strong epimorphism.

Indeed, start with a factorization of every f_A in \mathcal{A} as $SA \xrightarrow{e_A} RA \xrightarrow{m_A} TA$ with e_A split epic and m_A monic in \mathcal{A} . Then the diagonal fill-in makes R an endofunctor with natural transformations $e : S \rightarrow R$ and $m : R \rightarrow T$. The monad unit of R is $\mu^R = e \cdot \eta^S : Id \rightarrow R$. And the monad multiplication is given by the following diagonal fill-in:

$$\begin{array}{ccccc}
 SSA & \xrightarrow{e_A * e_A} & RRA & \xrightarrow{m_A * m_A} & TTA \\
 \downarrow \mu_A^S & & \downarrow \mu_A^R & & \downarrow \mu_A^T \\
 SA & \xrightarrow{e_A} & RA & \xrightarrow{m_A} & TA
 \end{array}$$

This is well-defined because $e_A * e_A = e_{RA} \cdot R e_A$ is a epimorphism. To verify the unit axioms $\mu^R \cdot \eta R = id$, consider the following diagram:

$$\begin{array}{ccccc}
 SA & \xrightarrow{e_A} & RA & \xrightarrow{m_A} & TA \\
 \downarrow \eta_{SA}^S & & \downarrow \eta_{RA}^R & & \downarrow \eta_{TA}^T \\
 SSA & \xrightarrow{e_A * e_A} & RRA & \xrightarrow{m_A * m_A} & TTA \\
 \downarrow \mu_A^S & & \downarrow \mu_A^R & & \downarrow \mu_A^T \\
 SA & \xrightarrow{e_A} & RA & \xrightarrow{m_A} & TA
 \end{array}$$

Its outward square commutes since S and T both satisfy the corresponding axiom. Naturality of η^S implies that the upper left-hand square commutes:

$$e_{RA} \cdot S e_A \cdot \eta_{SA}^S = e_{RA} \cdot \eta_{RA}^S \cdot e_A = \eta_A^R \cdot e_A.$$

Analogously for the upper right-hand square. Consequently, the diagonal passage from SA to TA in the above diagram satisfies (due to $\mu_A^T \cdot \eta_{TA}^T = id$) the equality

$$m_A \cdot (\mu_A^R \cdot \eta_{RA}^R) \cdot e_A = m_A \cdot e_A.$$

Since n_A is strongly monic and e_A epic, this implies $\mu_A^R \cdot \eta_{RA}^R = id$.

The verification of the other unit axiom $\mu^R \cdot R\eta^R = id$ is analogous.

The proof of the associativity

$$\mu^R \cdot R\mu^R = \mu^R \cdot \mu^R R$$

follows from the following diagram:

$$\begin{array}{ccccc}
 SSSA & \xrightarrow{e_A * e_A * e_A} & RRRA & \xrightarrow{m_A * m_A * m_A} & TTTA \\
 \mu_{SA}^S \downarrow & & \mu_{RA}^R \downarrow & & \mu_{TA}^T \downarrow \\
 SSA & \xrightarrow{e_A * e_A} & RRA & \xrightarrow{m_A * m_A} & TTA \\
 \mu_A^S \downarrow & & \mu_A^R \downarrow & & \mu_A^T \downarrow \\
 SA & \xrightarrow{e_A} & RA & \xrightarrow{m_A} & TA
 \end{array}$$

We only need to check that the epimorphism $e_A * e_A * e_A$ merges the above parallel pair. Since m_A is a monomorphism and the outward square of the above diagram is the following commutative square

$$\begin{array}{ccc}
 SSSA & \xrightarrow{f_A * f_A * f_A} & TTTA \\
 \mu_A^S \cdot S\mu_A^S \downarrow & & \mu_A^T \cdot T\mu_A^T \downarrow \\
 SA & \xrightarrow{f_A} & TA
 \end{array}$$

the associativity of μ^S and μ^T clearly implies that of μ^R . \square

Recall from Definition 2.5 the concept of arbitrarily large pre-fixpoints of an endofunctor. Here is a "collective" version:

Definition 3.4. A collection $F_i (i \in I)$ of endofunctors is said to have arbitrarily large joint pre-fixpoints if for every object A and every cardinal $\alpha > 0$ there exists a joint pre-fixpoint X such that $X + A \simeq X \simeq \coprod_{\alpha} X$.

Example 3.5. For categories **Set** and **K-Vec** or **Set_{*}** this means that for every cardinal α there exists a joint pre-fixpoint of cardinality at least α . (In **K-Vec** use the fact that for infinite cardinals $\alpha \geq \text{card}K$ dimension α is equivalent to cardinality α .)

For many-sorted sets, **Set^S**, this means that for every cardinal α there exists a joint pre-fixpoint whose components have cardinalities at least α .

Proposition 3.6. Every collection of accessible endofunctors has arbitrarily large joint pre-fixpoints.

PROOF. If $H_r (r \in R)$ are accessible endofunctors, then so is $H = \coprod_{r \in R} H_r$. And every pre-fixpoint of H is a joint pre-fixpoint of all H_r . Thus our task is for a given object A and an infinite cardinal α , to find a pre-fixpoint X of H with $X \simeq A + X \simeq \alpha \cdot X$. The copower $\alpha \bullet H$ of α copies of H is accessible, thus, it has a free algebra on $B = \alpha \bullet A$. As in the proof of Lemma 2.8 this free algebra \bar{B} fulfils

$$\bar{B} = \alpha \bullet A + \alpha \bullet H\bar{B} = \alpha \bullet (A + H\bar{B})$$

Obviously, $H\bar{B}$ is a subobject of $\alpha \bullet H\bar{B}$, hence, a pre-fixpoint of H . And $\bar{B} \simeq A + \bar{B} \simeq \alpha \bullet \bar{B}$. \square

Theorem 3.7. Every small collection of monos-preserving monads with arbitrarily large joint pre-fixpoints has a coproduct in **Monad** (\mathcal{A}).

PROOF. Let $S_i = (S_i, \mu_i, \eta_i), i \in I$, be such a collection. Then the endofunctor $S = \coprod_{i \in I} S_i$ preserves monomorphisms. And it has arbitrarily large pre-fixpoints: given an object A find X with $S_i X \rightarrowtail X$ for all $i \in I$ and $X \simeq X + A \simeq \coprod_I X$ to get

$$SX = \coprod_{i \in I} S_i X \rightarrowtail \coprod_I X \simeq X.$$

By Corollary 2.15 the functor $S = \coprod_{i \in I} S_i$ generates a free monad \mathbb{F}_S with the universal arrow $\hat{\eta} : S \rightarrow \mathbb{F}_S$; the coproduct injections are denoted by

$$v_i : S_i \rightarrow S \quad (i \in I)$$

The forgetful functor **Monad** (\mathcal{A}) $\rightarrow [\mathcal{A}, \mathcal{A}]$ creates limits, see Proposition 2.2, and we conclude that for the slice category $\mathbb{F}_S / \mathbf{Monad}(\mathcal{A})$ the corresponding forgetful functor

$$U : \mathbb{F}_S / \mathbf{Monad}(\mathcal{A}) \rightarrow \mathbb{F}_S / [A, A]$$

also creates limits. Now consider an arbitrarily cocone $f = (f_i)$ consisting of monad morphisms $f_i : \mathbb{S}_i \rightarrow \mathbb{T}_f$ ($i \in I$). The functor $[f_i] : S \rightarrow T_f$ generates uniquely a monad morphism $\bar{f} : \mathbb{F}_S \rightarrow \mathbb{T}_f$ with $\bar{f} \cdot \hat{\eta} = [f_i]$ that we factorize as in Lemma 3.3

$$\begin{array}{ccc}
 S & & \\
 \eta^S \downarrow & \searrow [f_i] & \\
 \mathbb{F}_S & \xrightarrow{\bar{f}} & \mathbb{T}_f \\
 e_f \searrow & & \nearrow m_f \\
 & \mathbb{R}_f &
 \end{array}$$

We get a (possibly large) collection of objects (e_f, R_f) of the slice category $\mathbb{F}_S / \text{Monad}(\mathcal{A})$. This collection has a product in $F_S / [\mathcal{A}, \mathcal{A}]$. Indeed, recall from Remark 3.2 that \mathcal{A} is cowellpowered, and for every object A form the meet of $(e_f)_A : F_S A \rightarrow R_f A$ ranging through all cocones f . Let $e_A : F_S A \rightarrow R A$ be meet, thus for every cocone f we have a morphism

$$q_f^A : R A \rightarrow R_f A \text{ with } (e_f)_A = q_f^A \cdot e_A.$$

The resulting functor R and natural transformations $q_f : R \rightarrow R_f$ form a product of all e_f in $F_S / [\mathcal{A}, \mathcal{A}]$. Consequently, there exists a product (e, R) of the objects (e_f, R_f) in $\mathbb{F}_S / \text{Monad}(\mathcal{A})$ as f ranges through all cocones: see Proposition 2.2. For the projections $q_f : R \rightarrow R_f$ define

$$p_f = m_f \cdot q_f : R \rightarrow \mathbb{T}_f.$$

Then $\bar{f} = m_f \cdot e_f = m_f \cdot q_f \cdot e = p_f \cdot e$ implies

$$f_i = f \cdot v_i = p_f \cdot e \cdot \hat{\eta} \cdot v_i :$$

$$(3.1) \quad
 \begin{array}{ccc}
 & S_i & \\
 & \downarrow v_i & \searrow f_i \\
 & S & \\
 & \downarrow \hat{\eta} & \\
 & \mathbb{S} & \\
 & \downarrow e & \searrow e_f \\
 u_i \swarrow & \mathbb{R} & \xrightarrow{p_f} \mathbb{T}_f
 \end{array}$$

We claim that \mathbb{R} is the coproduct of \mathbb{S}_i ($i \in I$) in $\text{Monad}(\mathcal{A})$ with respect to

$$u_i = e \cdot \hat{\eta} \cdot m_i : \mathbb{S}_i \rightarrow \mathbb{R} \quad (i \in I).$$

- (a) Each u_i is a monad morphism. This follows from the fact that (p_f) is a collectively monic cone in $[\mathcal{A}, \mathcal{A}]$ and each f_i is a monad morphism. Indeed, the condition $u_i \cdot \eta_i = \eta^R$ follows from

$$\begin{aligned} p_f \cdot (u_i \cdot \eta_i) &= f_i \cdot \eta_i && \text{see (3.1)} \\ &= \eta^{\mathbb{R}f} && f_i \text{ a monad morphism} \\ &= p_f \cdot \eta^{\mathbb{R}} && p_f \text{ a monad morphism.} \end{aligned}$$

The verification of the condition

$$\mu_i \cdot u_i = \mu^R \cdot u_i * u_i = \mu^R \cdot Ru_i \cdot u_i S_i$$

follows from the following diagram

$$\begin{array}{ccccc} S_i S_i & \xrightarrow{\mu_i} & & & S_i \\ & \searrow u_i S_i & & & \nearrow u_i \\ f_i S_i & & R S_i & & \\ & \searrow p_f S_i & \searrow R u_i & & \\ T_f S_i & & R R & \xrightarrow{\mu^R} & R \\ & \searrow T_f u_i & \downarrow p_f R & & \downarrow p_f \\ & \searrow T_f f_i & T_f R & & T_f \\ & & \downarrow T_f p_f & & \\ & & T_f T_f & \xrightarrow{\mu^{T_f}} & T_f \end{array}$$

All the inner parts but the upper one (to be proved commutative) commute: recall $f_i = p_f \cdot u_i$, use the fact that p_f is a monad morphism for the lower square, and use the naturality of p_f for $p_f R \cdot Ru_i = T_f u_i \cdot p_f S_i$. Since f_i is a monad morphism, the outward square also commutes. This, together with the collective monicity of all p_f 's, proves that the upper square commutes.

For every cocone $f = (f_i)_{i \in I}$ the monad morphism p_f is the desired factorization: $f_i = p_f \cdot u_i$, see (3.1). This is unique since whenever $r : \mathbb{R} \rightarrow \mathbb{T}_f$ is a monad morphism with $f_i = r \cdot u_i$ for all i , then $r \cdot e \cdot \hat{\eta} = f = p_f \cdot e \cdot \hat{\eta}$ which implies $r \cdot e = p_f \cdot e$ by the universal property of $\hat{\eta}$; hence $r = p_f$ since e is epic.

□

Remark 3.8. (a) *Kelly described colimits of monads, see [9], Section 27 as follows:*

Let D be a diagram in $\text{Monad}(\mathcal{A})$ with objects $\mathbb{T}_i = (T_i, \mu_i, \eta_i)$ for $i \in I$.

Form the category \mathcal{C}_D of all pairs $(A, (a_i)_{i \in I})$ where A is an object of \mathcal{A} and $a_i : T_i A \rightarrow A$ is an Eilenberg-Moore algebra for \mathbb{T}_i ($i \in I$) such that for every connecting morphism $f : i \rightarrow j$ of the indexing category the triangle

$$(3.2) \quad \begin{array}{ccc} T_i A & \xrightarrow{a_i} & A \\ & \searrow (Df)_A & \uparrow a_j \\ & & T_j A \end{array}$$

commutes. The morphisms of \mathcal{C}_D are the morphisms of \mathcal{A} which are algebra homomorphisms for every \mathbb{T}_i . We have the obvious forgetful functor

$$U_D : \mathcal{C}_D \rightarrow \mathcal{A}.$$

Kelly proved that if U_D has a left adjoint, then the corresponding monad on \mathcal{A} is a colimit of D in $\text{Monad}(\mathcal{A})$. The converse also holds if \mathcal{A} is a complete category.

Theorem 3.9. *Every diagram with a weakly terminal object has a colimit in $\text{Monad}(\mathcal{A})$. In particular, $\text{Monad}(\mathcal{A})$ has coequalizers.*

PROOF. Let $D : \mathcal{D} \rightarrow \text{Monad}(\mathcal{A})$ be a diagram with objects $\mathbb{T}_i = (T_i, \mu_i, \eta_i)$ for $i \in I$, and let \mathbb{T}_j be weakly terminal, i.e., for every $i \in I$ there exists a connecting morphism $f : \mathbb{T}_i \rightarrow \mathbb{T}_j$ in D .

- (a) Form the full subcategory \mathcal{C} of $\mathcal{A}^{\mathbb{T}_j}$ of all algebras $a : T_j A \rightarrow A$ for \mathbb{T}_j such that for every pair $f, g : \mathbb{T}_i \rightarrow \mathbb{T}_j$ of connecting morphisms of D ($i \in I$) we have

$$(3.3) \quad a \cdot f_A = a \cdot g_A$$

This category is closed in $\mathcal{A}^{\mathbb{T}_j}$ under products, which easily follows from the forgetful functor $U^{\mathbb{T}_j}$ creating limits. It is also closed under subalgebras. More precisely, let $m : (A, a) \rightarrow (B, b)$ be a homomorphism in $\mathcal{A}^{\mathbb{T}_j}$ with m

monic in \mathcal{A} . If (B, b) lies in \mathcal{C} , then so does (A, a) :

$$\begin{array}{ccccc}
 T_i A & \xrightleftharpoons[f_A]{g_A} & T_j A & \xrightarrow{a} & A \\
 \downarrow T_i m & & \downarrow T_j m & & \downarrow m \\
 T_i B & \xrightleftharpoons[f_B]{g_B} & T_j B & \xrightarrow{b} & B
 \end{array}$$

Since the forgetful functor $U^{\mathbb{T}_j}$ creates limits, the category $\mathcal{A}^{\mathbb{T}_j}$ is complete and wellpowered. Let us prove that it is also cowellpowered. Given a factorization of a homomorphism $h : (A, a) \rightarrow (B, b)$ in $\mathcal{A}^{\mathbb{T}_j}$ as a strong epimorphism $e : C \rightarrow B$ followed by a monomorphism $m : C \rightarrow B$ in \mathcal{A} , the diagonal fill-in makes e and m homomorphisms:

$$\begin{array}{ccc}
 T_j A & \xrightarrow{a} & A \\
 T_j e \downarrow & & \downarrow e \\
 T_j C & \xrightarrow{c} & C \\
 T_j m \downarrow & & \downarrow m \\
 T_j B & \xrightarrow{b} & B
 \end{array}$$

Thus, if h is a strong epimorphism in $\mathcal{A}^{\mathbb{T}_j}$ then m is an isomorphism (recall that $U^{\mathbb{T}_j}$ creates limits, thus, reflects isomorphisms), consequently, h is an epimorphism in \mathcal{A} . Since \mathcal{A} is cowellpowered (see Remark 3.2) we conclude that $\mathcal{A}^{\mathbb{T}_j}$ is cowellpowered.

- (b) Every full subcategory of $\mathcal{A}^{\mathbb{T}_j}$ closed under products and subobjects is reflective, see [4], 16.9. Thus, the obvious forgetful functor $U : \mathcal{C} \rightarrow \mathcal{A}$ has a left adjoint.

The theorem now follows from Remark 3.8 and the fact that there exists an isomorphism E of categories such that the triangle

$$\begin{array}{ccc}
 \mathcal{C}_D & \xrightarrow{E} & \mathcal{C} \\
 & \searrow U_D & \swarrow U \\
 & \mathcal{A} &
 \end{array}$$

commutes. Indeed, E is the "projection to j "

$$E(A, (d_i)_{i \in I}) = (A, d_j).$$

From the triangles (3.2) we deduce that (A, d_j) satisfies (3.3). Thus, E is a well-defined, faithful functor. It is surjective on objects: for every algebra (A, a) in \mathcal{C} define, given $i \in I$,

$$a_i = a \cdot f_A : T_i A \rightarrow A \text{ for any connecting morphism } f : T_i \rightarrow T_j.$$

Then a_i is well-defined due to (3.3) and, since f is a monad morphism, (A, a_i) is an Eilenberg-Moore algebra for T_i . Finally, to prove that E is an isomorphism, we verify that it is full. Let

$$k : (A, a) \rightarrow (B, b)$$

be a homomorphism in \mathcal{C} . Then we need to prove that for every $i \in I$ this is a homomorphism from (A, a_i) to (B, b_i) , where again $b_i = b \cdot f_B$. Use the following diagram

$$\begin{array}{ccccc}
 & & a_i & & \\
 & \frown & & \smile & \\
 T_i A & \xrightarrow{f_A} & T_j A & \xrightarrow{a} & A \\
 \downarrow T_i k & & \downarrow T_j k & & \downarrow k \\
 T_i B & \xrightarrow{f_B} & T_j B & \xrightarrow{b} & B \\
 & \smile & & \frown & \\
 & & b_i & &
 \end{array}$$

□

Corollary 3.10. *Every diagram of monos-preserving monads with arbitrarily large joint pre-fixpoints has a colimit in $\text{Monad}(\mathcal{A})$.*

Indeed, apply the usual construction of colimits as coequalizers of a parallel pair between coproducts; see [12]. Given a diagram \mathcal{D} in $\text{Monad}(\mathcal{A})$ with monos-preserving objects $\mathbb{S}_i = (S_i, \mu_i, \eta_i)$ for $i \in I$ having arbitrarily large joint pre-fixpoints, then also every collection of monads indexed by $I \times J$, where J is an arbitrarily set and $\mathbb{S}_i = \mathbb{S}_{(i,j)}$ for all $(i,j) \in I \times J$, has arbitrarily large joint pre-fixpoint. (Indeed, for every object A and every cardinal α put $\alpha' = \alpha + \text{card} J$. By applying Definition 3.4 to A and α' for the former collection indexed by I , we get the required condition for the new collection.) Thus, those two coproducts needed to construct $\text{colim } \mathcal{D}$ as a coequalizer in $\text{Monad}(\mathcal{A})$ exist.

Remark 3.11. *Monad (\mathcal{A}) also has cointersections. That is, wide pushouts of strong epimorphisms $e_i : \mathbb{T} \rightarrow \mathbb{S}_i$ ($i \in I$). The proof is analogous to that of Theorem 3.9. Let \mathcal{C} be the full subcategory of $\mathcal{A}^{\mathbb{T}}$ on all algebras $a : TA \rightarrow A$ for which a factorized through each $(e_i)_A : TA \rightarrow \mathbb{S}_i A$ factorizes through a . This subcategory is easily seen to be closed under products and subalgebras. And it is isomorphic to the category $\mathcal{C}_{\mathcal{D}}$ of Remark 3.8. (Here we use the fact established in Lemma 3.3 that strong epimorphisms in $\text{Monad}(\mathcal{A})$ have epic components.) Thus, the cointersection of c_i exists in $\text{Monad}(\mathcal{A})$.*

Example 3.12. *For the base category of graphs*

$$\mathbf{Gra} = \mathbf{Set}^{\rightrightarrows}$$

we present a parallel pair of monad morphisms having no coequalizer in $\text{Monad}(\mathbf{Gra})$.

For every graph $X = (V, E, s, t)$ with source and target maps $s, t : E \rightarrow V$ we denote by X_e the set of all loops, i. e., the equalizer of s and t . We construct two endofunctors $H, K : \mathbf{Gra} \rightarrow \mathbf{Gra}$ and two natural transformations $\sigma, \tau : H \rightarrow K$ such that for the coequalizer

$$H \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{\tau} \end{array} K \xrightarrow{\rho} L \quad \text{in } [\mathbf{Gra}, \mathbf{Gra}]$$

L does not generate a free monad, but H and K do. It follows immediately that the monad morphisms

$$\bar{\sigma}, \bar{\tau} : \mathbb{F}_H \rightarrow \mathbb{F}_K$$

corresponding to σ and τ do not have a coequalizer in $\text{Monad}(\mathbf{Gra})$: if \mathbb{S} were the codomain of such a coequalizer, then since $\mathbb{F}_{(-)}$ is a left adjoint, \mathbb{S} would clearly be a free monad on L .

Let \mathcal{P} denote the power-set functor. The endofunctor H is defined on objects X as follows:

$H(X)$ has vertices $\mathcal{P}(X_e)$ and no edges.

The definition of H on morphisms $g : X \rightarrow X'$ is as expected: $H(g)$ is the domain-codomain restriction of the edge function of g to all loops. Analogously define K :

$K(X)$ has vertices $\mathcal{P}(X_e) + \mathcal{P}(X_e)$ and edges $\mathcal{P}(X_e)$

$s, t : \mathcal{P}(X_e) \rightarrow \mathcal{P}(X_e) + \mathcal{P}(X_e)$ are the coproduct injections

That is, $K(X)$ is the disjoint union of arrows indexed by $\mathcal{P}(X_e)$. The definition on morphisms is again as expected. Let

$$\sigma, \tau : H \rightarrow K$$

be the natural transformations corresponding to s and t : for every $M \subseteq X_e$, $\sigma_X(M)$ is the source of the arrow labelled by M and $\tau_X(M)$ is its target. The coequalizer L of σ and τ in $[\mathbf{Gra}, \mathbf{Gra}]$ is obvious: it assigns to every graph X the graph on $\mathcal{P}(X_e)$ consisting of loops:

$$L(X) \text{ has vertices} = \text{edges} = \mathcal{P}(X_e) \text{ and } s = t$$

The functor H generates a free monad, since in Construction 2.11 we have

$$W_2 = X + H(X + HX) = H + HX = W_1.$$

Thus the construction converges in one step. The same is true about K .

It remains to prove that L does not generate a free monad. By Theorem 2.9 it is sufficient to prove that L does not have an initial algebra. Indeed, we prove that if

$$a : LA \rightarrow A$$

is an initial algebra, then \mathcal{P} has an initial algebra (compare Example 2.10). Let $m : A_0 \rightarrow A$ be the subgraph of A whose vertices are precisely the loops of A and whose edges are just all the loops. Then $LA = LA_0$, and we obviously have a codomain restriction $a_0 : LA_0 \rightarrow A_0$ of a . And $m : (A_0, a_0) \rightarrow (A, a)$ is a homomorphism of algebras for L . The unique homomorphism $h : (A, a) \rightarrow (A_0, a_0)$ thus yields an endomorphism $m \cdot h$ of the initial algebra; hence $m \cdot h = \text{id}$. This proves $A = A_0$. That is, A is the set A_v of vertices endowed with all loops. But then $a : \mathcal{P}A_v \rightarrow A_v$ as an algebra for \mathcal{P} is initial: given any algebra $b : \mathcal{P}B \rightarrow B$, form the graph \bar{B} of all loops in B and obtain an obvious structure $\bar{b} : L\bar{B} \rightarrow \bar{B}$ of an L -algebra. Then \mathcal{P} -algebra homomorphisms from (A, a) to (\bar{B}, \bar{b}) are precisely the L -algebra homomorphisms from (A_v, a) to \bar{B} . This is the desired contradiction.

Example 3.13. The category $\mathbf{Monad}(\mathbf{Gra})$ also fails to have cointersections of split epimorphisms. The argument is completely analogous: the following split epimorphisms

$$\sigma_0 = [\sigma, \sigma, \tau, \text{id}] : H + H + H + K \rightarrow K$$

and

$$\tau_0 = [\tau, \sigma, \tau, \text{id}] : H + H + H + K \rightarrow K$$

of $[\mathbf{Gra}, \mathbf{Gra}]$ have the cointersection as follows:

$$\begin{array}{ccc}
 & \hat{H} = H + H + H + K & \\
 \sigma_0 \swarrow & & \searrow \tau_0 \\
 K & & K \\
 \rho \searrow & & \swarrow \rho \\
 & L &
 \end{array}$$

Since L does not generate a free monad, the split epimorphisms $\bar{\sigma}_0, \bar{\tau}_0 : \mathbb{F}_{\hat{H}} \rightarrow \mathbb{F}_K$ do not have a cointersection in $\mathbf{Monad}(\mathbf{Gra})$.

4. COPRODUCTS OF SEPARATED MONADS

Ghani and Ustalu presented in [8] an interesting formula for coproducts of ideal monads, see Example 4.6(4), which was, in case of monads over \mathbf{Set} , generalized in [2]. The present section is based on the ideas of the latter paper, extending the formula to separated monads over abstract categories. Separateness means that the monad unit has a complement – not over the given category \mathcal{A} but over the category \mathcal{A}_m of all objects and all monomorphisms.

Assumption 4.1. *Throughout this section \mathcal{A} denotes a cocomplete category in which a coproduct of parallel monomorphisms is always monic.*

We denote by

$$\mathcal{A}_m$$

the category of all objects and all monomorphisms of \mathcal{A} .

Every monos-preserving endofunctor F of \mathcal{A} defines an endofunctor of \mathcal{A}_m by restriction, we denote it by F again.

The coproduct $+$ of \mathcal{A} is a monoid structure on \mathcal{A}_m (not having the universal property of coproducts, of course).

- Example 4.2.** (1) The exception monad \mathbb{M}_E defined by $X \mapsto X + E$ has coproduct with all monads \mathbb{S} : the coproduct is given by $X \mapsto S(X + E)$.
(2) The terminal monad $\mathbb{1}$ given by $X \mapsto 1$, also has all coproducts, the result is always $\mathbb{1}$.
(3) For monads over \mathbf{Set} there are essentially no other monads having a coproduct with every monad. More precisely, let \mathbb{M}_E^0 be the modification of \mathbb{M}_E with $\emptyset \mapsto \emptyset$ and $X \mapsto E$ for all $X \neq \emptyset$. Analogously, let $\mathbb{1}^0$ be given by $\emptyset \mapsto \emptyset$ and $X \mapsto 1$ for all $X \neq \emptyset$. It is easy to see that $\mathbf{Monad}(\mathbf{Set})$ has all coproducts with \mathbb{M}_E^0 or with $\mathbb{1}^0$.

Definition 4.3. We call a monad over **Set** trivial if it is isomorphic to \mathbf{M}_E , \mathbf{M}_E^0 , $\mathbf{1}$, or $\mathbf{1}^0$. These are precisely the monads corresponding to varieties of algebras with no operation of arity at least 1.

Theorem 4.4 (See [2]). A monad over **Set** has coproducts with all monads iff it is trivial.

Moreover, all monads over **Set** except $\mathbf{1}$ and $\mathbf{1}^0$ are consistent, i. e., the components of the monad unit are monic.

4.1. The category of multi-algebras. Given a discrete diagram \mathcal{D} of monads \mathbb{T}_i ($i \in I$) the category $\mathcal{C}_{\mathcal{D}}$ of Remark 3.8 has as objects *multi-algebras*

$$(A, (a_i)_{i \in I}) \text{ where } a_i : T_i A \rightarrow A \text{ lies in } \mathcal{A}^{\mathbb{T}_i}$$

and morphisms are those maps in \mathcal{A} that are homomorphisms for each of \mathbb{T}_i simultaneously. A coproduct of the monads \mathbb{T}_i exists in $\text{Monad}(\mathcal{A})$ whenever every object of \mathcal{A} generates a free multi-algebra.

Definition 4.5. A monad (S, μ, η) is called separated if its unit has a complement in the following sense:

(i) S preserves monomorphisms

and

(ii) there exists an endofunctor \bar{S} of \mathcal{A}_m such that

$$S = Id + \bar{S}$$

with the unit η as the left-hand injection.

Examples 4.6. (1) The exception monad \mathbf{M}_E is separated: here \bar{M}_E is the constant functor of value E .

(2) Every free monad \mathbb{F}_H which preserves monomorphisms is separated. (In particular, if \mathcal{A} has stable monomorphisms, all free monads on monos-preserving functor are separated.) Here $\bar{\mathbb{F}}_H = H \cdot \mathbb{F}_H$: use Remarks 2.16 and 2.17.

(3) All consistent monads on **Set** (i. e., all except $\mathbf{1}$ and $\mathbf{1}^0$) are separated. See [2], Proposition IV.5.

(4) Ideal monads of Elgot [7] are separated if they preserve monomorphisms. Recall that an ideal monad $\mathbb{S} = (S, \mu, \eta)$ is one for which an endofunctor \bar{S} of \mathcal{A} exists such that (i) $S = Id + \bar{S}$ in $[\mathcal{A}, \mathcal{A}]$ with the left-hand injection η and (ii) μ restricts to a natural transformation $\bar{\mu} : \bar{S}S \rightarrow \bar{S}$.

(5) In particular, the free completely iterative monad \mathbb{S} on an endofunctor H given by the greatest fixpoint

$$SA = \nu X \cdot (A + HX)$$

is separated, with $\bar{S} = H \cdot S$, whenever it preserves monomorphisms, see [3].

Notation 4.7. Let \mathbb{S}_i ($i \in I$) be separated monads. For every object A of \mathcal{A} define an endofunctor H_A of \mathcal{A}_m^I as follows:

$$H_A(X_i)_{i \in I} = (\bar{S}_i Y_i)_{i \in I} \text{ where } Y_i = A + \coprod_{j \in I, j \neq i} X_j$$

If H_A has an initial algebra, we denote its components by $S_i^* A$:

$$\mu H_A = (S_i^* A)_{i \in I}$$

Remark 4.8. Let (X_i) be a fixed point of H_A :

$$X_i \simeq \bar{S}_i Y_i \text{ for all } i \in I$$

Then the coproduct $A + \coprod_{i \in I} X_i$ carries a canonical structure of a multi-algebra: the algebra structure for \mathbb{S}_i is the free algebra on Y_i . Indeed, the usual free algebra is $(S_i Y_i, \mu_{Y_i}^i)$. And the above coproduct is isomorphic to $S_i Y_i$:

$$\begin{aligned} A + \coprod_{i \in I} X_i &\cong Y_i + X_i \\ &\simeq Y_i + \bar{S}_i Y_i \\ &= S_i Y_i \end{aligned}$$

In particular: if the initial algebra $\mu H_A = (S_i^* A)_{i \in I}$ exists, then the coproduct

$$A + \coprod_{i \in I} S_i^* A$$

is a multi-algebra. We prove that it is free on A w.r.t. the right-hand coproduct injection $\text{inl} : A \rightarrow A + \coprod_{i \in I} S_i^* A$:

Theorem 4.9. A coproduct of separated monads \mathbb{S}_i ($i \in I$) exists whenever the initial algebra $\mu H_A = (S_i^* A)$ exists for every object A . It is defined by

$$A \mapsto A + \coprod_{i \in I} S_i^* A$$

Remark 4.10. The monad unit η_A is the right-hand coproduct injection. The multiplication follows from $A + \coprod_{i \in I} S_i^* A$ being the free multi-algebra on A .

PROOF. Let $\mathbb{S}_i = (S_i, \mu^i, \eta^i)$ be the given monads. Following Remark 3.8 all we need proving is that the multi-algebra $\bar{A} = A + \coprod_{i \in I} S_i^* A$ is free.

(1) Let us describe its algebra structure explicitly for every \mathbb{S}_i . The initial-algebra structure of μH_A is given by isomorphisms

$$\varphi_i : \bar{S}_i Y_i \rightarrow X_i$$

where $X_i = S_i^* A$ and $Y_i = A + \coprod_{j \neq i} X_j$. This defines isomorphisms

$$(4.4) \quad \bar{\varphi}_i \equiv \bar{A} = Y_i + X_i \xrightarrow{Y_i + \varphi_i^{-1}} Y_i + \bar{S}_i Y_i = S_i Y_i$$

And the algebra structure σ_i of \bar{A} for \mathbb{S}_i is transported by this isomorphism from the free-algebra structure $\mu_{Y_i}^i$:

$$\begin{array}{ccc} S_i \bar{A} & \xrightarrow{\sigma_i} & \bar{A} \\ S_i \bar{\varphi}_i \downarrow & & \uparrow \bar{\varphi}_i^{-1} \\ S_i S_i Y_i & \xrightarrow{\mu_{Y_i}^i} & S_i Y_i \end{array}$$

(2) For every multi-algebra

$$\beta_i : S_i B \rightarrow B \quad (i \in I)$$

and every morphism $f : A \rightarrow B$ we prove that a unique multi-algebra homomorphism

$$\bar{f} : \bar{A} \rightarrow B \text{ with } f = \bar{f} \cdot \text{inl}$$

exists. The object $\nabla B = (B, B, B \dots)$ is an algebra for H_A w.r.t. $(b_i)_{i \in I} : H_A(\nabla B) \rightarrow \nabla B$ given as follows:

$$b_i \equiv \bar{S}_i(A + \coprod_{I - \{i\}} B) \xrightarrow{\bar{S}_i[f, \nabla]} \bar{S}_i B \hookrightarrow S_i B \xrightarrow{\beta_i} B$$

The middle subobject is the right-hand coproduct injection of $S_i B = B + \bar{S}_i B$. We have a unique homomorphism from the initial algebra μH_A :

$$(h_i)_{i \in I} : (X_i)_{i \in I} \rightarrow \nabla B$$

which means that the square

$$(4.5) \quad \begin{array}{ccc} \bar{S}_i Y_i = \bar{S}_i(A + \coprod_{j \neq i} X_j) & \xrightarrow{\varphi_i} & X_i \\ \bar{S}_i(A + \coprod_{j \neq i} h_j) \downarrow & & \downarrow h_i \\ \bar{S}_i(A + \coprod_{j \neq i} B) & \xrightarrow{b_i} & B \end{array}$$

commutes for every i . Put

$$\bar{h}_i = [h_j]_{j \neq i} : \coprod_{j \in I, j \neq i} X_j \rightarrow B$$

Then (4.5) is equivalent to the commutativity of the following square:

$$(4.6) \quad \begin{array}{ccc} \bar{S}_i Y_i & \xrightarrow{\varphi_i} & X_i \\ \bar{S}_i[f, \bar{h}_i] \downarrow & & \downarrow h_i \\ \bar{S}_i B & \xrightarrow{\quad} S_i B & \xrightarrow{\beta_i} B \end{array}$$

We are going to prove that the desired extension of f is

$$\bar{f} = [f, \bar{h}] : A + \coprod_{j \in I} X_j \rightarrow B \text{ where } \bar{h} = [h_j]_{j \in I}.$$

That is, we first need to prove that \bar{f} is a homomorphism for H_A . Thus for every $i \in I$ we must prove that the following diagram commutes:

$$(4.7) \quad \begin{array}{ccccccc} S_i \bar{A} & \xrightarrow{S_i \bar{\varphi}_i} & S_i S_i Y_i & \xrightarrow{\mu_Y^i} & S_i Y_i & \xrightarrow{\bar{\varphi}_i^{-1}} & \bar{A} = A + \coprod_{j \neq i} X_j \\ \downarrow S_i \bar{f} & & \downarrow S_i S_i \bar{f} & & \downarrow S_i \bar{f} & & \downarrow \bar{f} \\ & & S_i S_i B & \xrightarrow{\mu_B^i} & S_i B & & \\ & \swarrow S_i \beta_i & & & \searrow \beta_i & & \\ S_i B & & & & & & B \\ & \xrightarrow{\beta_i} & & & & & \end{array}$$

(The upper line is the algebra structure σ_i of \bar{A} .) The middle square is the naturality of μ^i , the lower-one is a monad-algebra axiom for (B, β_i) . We only need to prove that the right-hand square commutes: the left-hand one is its image under S_i . Using $S_i Y_i = Y_i + \bar{S}_i Y_i$ we get the following presentation of the right-hand square, recalling (4.4):

$$\begin{array}{ccc} S_i Y_i = Y_i + \bar{S}_i Y_i & \xrightarrow{Y_i + \varphi_i} & A + \coprod_{j \neq i} X_j \\ \bar{S}_i[f, [\bar{h}_i]] + [f, [\bar{h}_i]] \downarrow & & \downarrow [f, [\bar{h}_i]] \\ B + \bar{S}_i B & \xrightarrow{\beta_i} & B \end{array}$$

The left-hand component with domain Y_i clearly commutes: recall that $\eta_B^i = \text{inl} : B \rightarrow S_i B$, thus $\beta_i \cdot \text{inl} = \text{id}$ due to the monad axioms for (B, β_i) . The right-hand component forms the square (4.6).

- (3) To prove uniqueness, let $\bar{f} : \bar{A} \rightarrow B$ be a multi-algebra homomorphism with $\bar{f} \cdot \text{inl} = f$. Define $h_i : X_i \rightarrow B$ to be the i -th component of \bar{f} , thus,

$\bar{f} = [f, [\bar{h}_i]]$. It is only needed to prove that the squares (4.5) commute: then \bar{h}_i 's are determined uniquely, since (X_i) is the initial algebra of H_A . Since \bar{f} is a multi-algebra homomorphism, (4.6) commutes. This clearly implies that (4.5) does. \square

Theorem 4.11 (See [2]). *For monads over \mathbf{Set} the above sufficient condition for coproducts is essentially necessary: a coproduct of separated (= consistent) monads exists iff*

(a) *for every set A the initial algebra of H_A exists*

or

(b) *all but one of the monads is trivial (i. e. isomorphic to \mathbb{M}_E or \mathbb{M}_E^0).*

Corollary 4.12. *Let \mathcal{A} have stable monomorphisms. Every collection of separated monads with arbitrarily large joint pre-fixpoints has a coproduct in $\mathbf{Monad}(\mathcal{A})$.*

Indeed, assuming $\mathbb{S}_i, i \in I$, have arbitrarily large joint pre-fixpoints, we prove that the endofunctor H_A has an initial algebra. By Corollary 2.15 we only need to find, for every object $X = (X_i)$ of \mathcal{A}_m^I , a prefixed point Z of H_A with $Z \simeq Z + X$.

The functor $S = \coprod_I \coprod_{i \in I} S_i$ has arbitrarily large pre-fixpoints: given an object Y of \mathcal{A} , let V be a joint pre-fixpoint of all S_i with $Y + V \simeq V \simeq \coprod_{I+I} V$, then V is a pre-fixpoint of S due to

$$SV = \coprod_I \coprod_{i \in I} S_i V \hookrightarrow \coprod_I \coprod_I V \simeq V.$$

By Corollary 2.15, S has a free algebra on

$$Y = A + \coprod_{\mathbb{N}} \coprod_{i \in I} X_i$$

(for the above object X of \mathcal{A}^I). Put $Y^* = F_S Y$. Remark 2.17 yields

$$Y^* = SY^* + Y = SY^* + A + \coprod_{\mathbb{N}} \coprod_{i \in I} X_i.$$

The desired object Z of \mathcal{A}_m^I is $Z = (Y, Y, Y, \dots)$. Obviously $Y_i \cong Y_i + X_i$, thus, $Z \simeq Z + X$. And $H_A Z = (A + \coprod_{j \neq i} \bar{S}_j Y^*)_{i \in I}$ is a subobject of Z due to the following monomorphism:

$$A + \coprod_{j \neq i} \bar{S}_j Y^* \hookrightarrow Y + \coprod_I SY^* \simeq Y + SY^* \simeq Y^*.$$

Corollary 4.13. *Let \mathcal{A} have stable monomorphisms. A coproduct of accessible separated monads \mathbb{S} and \mathbb{T} is given by*

$$A \mapsto A + \operatorname{colim} X_k + \operatorname{colim} Y_k$$

for the transfinite chains

$$X_k : 0 \rightarrow \bar{S}A \rightarrow \bar{S}(A + \bar{T}A) \rightarrow \dots$$

and

$$Y_k : 0 \rightarrow \bar{T}A \rightarrow \bar{T}(A + \bar{S}A) \rightarrow \dots$$

More precisely, these chains are defined by the mutual recursion

$$(4.8) \quad X_{k+1} = \bar{S}(A + Y_k) \text{ and } Y_{k+1} = \bar{T}(A + X_k)$$

on isolated steps, and by colimits on limit steps.

To see this, let λ be an infinite cardinal such that S and T preserve λ -filtered colimits. Then \bar{S} and \bar{T} also preserve λ -filtered colimits. (Indeed, given a λ -filtered colimit $b_j : B_j \rightarrow B, j \in J$, we know that $Sb_j = b_j + \bar{S}b_j$ is also a colimit cocone. For every cocone $c_j : \bar{S}B_j \rightarrow C$ consider the cocone $b_j + c_j : SB_j \rightarrow B + C$. Since this factorizes uniquely through Sb_j , it follows that c_j factorizes uniquely through $\bar{S}b_j$. Thus \bar{S} preserves λ -filtered colimits, analogously \bar{T}). Consequently, the functor $H_A(V, W) = (\bar{S}(W + A), \bar{T}(V + A))$ preserves λ -filtered colimits. This implies, as proved in [A], that μH_A is the colimit of the λ -chain (X_i, Y_i) which is the free-algebra chain H_A and the initial object X of \mathcal{A}_m^I , see Construction 2.11. The recursion $W_{k+1} = H_A W_k$ is precisely (4.8) above.

Remark 4.14. *More generally, a coproduct of accessible separated monads $\mathbb{S}_i (i \in I)$ is given by*

$$A \mapsto A + \coprod_{i \in I} X_k^i$$

for the transfinite $(X_k^i)_{k \in \mathbf{Ord}}$ chains given on isolated steps by

$$X_{k+1}^i = \bar{S}_i(A + \coprod_{j \neq i} X_k^j)$$

and on limit steps by colimits.

Notation 4.15. *For a separated monad \mathbb{S} define endofunctors \bar{S}_A of \mathcal{A}_m by*

$$\bar{S}_A X = \bar{S}(A + X).$$

Thus, the above formula simplifies to $X_{k+1}^i = (\bar{S}_i)_A \coprod_{j \neq i} X_k^j$. For two monads we also have a more compact formula:

Corollary 4.16. *Let \mathcal{A} have stable monomorphisms. The coproduct of a pair \mathbb{S}, \mathbb{T} of separated monads with arbitrarily large joint pre-fixpoints is given by*

$$A \mapsto A + \mu \bar{S}_A \bar{T}_A + \mu \bar{T}_A \bar{S}_A.$$

Indeed, the coproduct is given by $A + S^*A + T^*A$, so all we need proving is that the endofunctor H_A has the initial algebra carried by $(\mu \bar{S}_A \bar{T}_A, \mu \bar{T}_A \bar{S}_A)$. We prove a more general statement:

Lemma 4.17. *Given endofunctors F and G of \mathcal{A} define an endofunctor H of \mathcal{A}^2 by $H(V, W) = (FW, GV)$. If (X, Y) is an initial algebra of H , then $X = \mu FG$ and $Y = \mu GF$.*

PROOF. Let the algebra structure of $\mu H = (X, Y)$ be given by

$$x : FY \xrightarrow{\sim} X \text{ and } y : GY \xrightarrow{\sim} X.$$

Then we prove that GF has the initial algebra

$$GFY \xrightarrow{Gx} GX \xrightarrow{y} X,$$

by symmetry $\mu FG = X$.

For every algebra $\beta : GFB \rightarrow B$ of GF form the algebra for H on (FB, B) with the following structure

$$id : FB \rightarrow FB \text{ and } \beta : GFB \rightarrow B.$$

Given the unique homomorphism of H -algebras

$$(a, b) : (X, Y) \rightarrow (FB, B)$$

it is easy to verify that $b : (X, y.Gx) \rightarrow (B, \beta)$ is a homomorphism for GF . Conversely, if $b : (X, y.Gx) \rightarrow (B, \beta)$ is a homomorphism for GF , then put $a = Fb.x^{-1} : X \rightarrow FB$. Then $(a, b) : (X, Y) \rightarrow (FB, B)$ is a homomorphism for H . Thus, b is the unique homomorphism for GF , proving $\mu GH = X$. \square

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